

On Inverse Surfaces in Euclidean 3-Space

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Abstract. In this paper, we study the inverse surfaces in 3-dimensional Euclidean space \mathbb{E}^3 . We obtain some results relating Christoffel symbols, the normal curvatures, the shape operators and the third fundamental forms of the inverse surfaces.

Keywords. Inversion, inversion curve, regular and singular points, Christoffel symbols.

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1. Introduction

The inversive geometry have been studied for many years by various mathematicians. See for instance, [2, 4, 5, 6, 9, 10, 11, 13]. In [3, 8, 12], the authors gave inversions of curves and surfaces. In [1], he obtained some results relating inverse surfaces.

During the same time that projective geometry was emerging, mathematicians mostly began to deal with circles. Their key tool was inversion. This interest gave rise to a new geometry, called inversive geometry, which was able to provide particularly striking proofs of previously known results in Euclidean geometry as well as new results.

This geometry have a wide application area in physics. For example, in the study of electrostatics let l_1 and l_2 be two infinitely long parallel cylinders of opposite charge. Then intersection of the surface of equipotential with a horizontal plane is two families of circles (and a single line), and a point charge placed in the electrostatic field moves along a circular path through a specific point inside each cylinder, at right angles to circles in the families. This geometry deal with such families of circles.

Now let put an inversion. Let S be a sphere with center c and radius r . Suppose that point p is other than c , then the inverse of p with respect to the sphere S is the point q on line cp such that

$$\|\vec{cp}\| \|\vec{cq}\| = r^2. \quad (1.1)$$

The sphere S is called the sphere of inversion, the point c is the center of inversion and the radius r is the radius of inversion.

The inversion maps the inside of sphere into outside vice versa and it unchange points on the sphere. c has no image, and no point of the space is mapped to c . But, the points close to c are mapped to points far from c and vice versa. So, let us say that the image of c "point at infinity" and vice versa.

In this study is organised as follows: Initially, it was obtained that regularity and singularity for inverse surfaces are properties remaining unchanged under an inversion. Then, some relations concerning the the normal curvatures, the shape operators, Christoffel symbols, the asymptotic and the conjugate vectors of inverse surfaces were found.

2. Basic Notions

Let $c \in \mathbb{E}^n$ and $r \in \mathbb{R}^+$. We denote that $\mathbb{E}^{n*} = \mathbb{E}^n - \{c\}$. Then, an inversion of \mathbb{E}^n with the center $c \in \mathbb{E}^n$ and the radius r is the map

$$inversion [c, r] : \mathbb{E}^{n*} \longrightarrow \mathbb{E}^{n*}$$

given by

$$inversion [c, r] (p) = c + \frac{r^2}{\|p - c\|^2} (p - c). \quad (2.1)$$

We use $inv [c, r]$ instead of $inversion [c, r]$ for shortness.

The inversion of a curve $\alpha : (a, b) \longrightarrow \mathbb{E}^{n*}$ is the curve

$$t \longrightarrow inversioncurve [c, r, \alpha] (t)$$

defined by

$$inversioncurve [c, r, \alpha] (t) = inv [c, r] (\alpha (t)). \quad (2.2)$$

Figure 1: Astroid and its inversion curve with respect to unit circle.

As shown in Figure 1, the points of astroid which are exterior to circle of inversion are mapped to points on the interior circle of inversion, and vice versa.

Let $x : U \longrightarrow \mathbb{E}^{n*}$ be a patch. The inverse patch of x with respect to $inv [c, r]$ is the patch given by

$$y = inv [c, r] \circ x. \quad (2.3)$$

Figure 2: Circular cylinder and its inverse patch.

As shown above the figure, the points of the cylinder which are exterior to the sphere of inversion are mapped to the points on interior, and vice versa. Note that as the cylinder arise in direction $\pm z$ its inverse close to center of inversion (origin).

Throughout this paper, we assume that Φ is $inv [c, r]$ and

$$y = \Phi \circ x.$$

Here $x : U \longrightarrow \mathbb{E}^{3*}$ is a patch in \mathbb{E}^{3*} .

Definition 2.1. ([7]) Let $\Phi = inv [c, r]$. Then, the tangent map of Φ at p is the map

$$\Phi_{*p} = T_p \left(\mathbb{E}^{n*} \right) \longrightarrow T_{\Phi(p)} \left(\mathbb{E}^{n*} \right)$$

given by

$$\Phi_{*p} (v_p) = \frac{r^2 v_p}{\|p - c\|^2} - \frac{2r^2 \langle (p - c), v_p \rangle}{\|p - c\|^4} (p - c), \quad (2.4)$$

where $v_p \in T_p \left(\mathbb{E}^{n*} \right)$.

Lemma 2.2. ([7]) An inversion is a conformal diffeomorphism.

3. The Shape Operators and Christoffel Symbols of Inverse Surfaces

Let $x : U \rightarrow \mathbb{E}^{3*}$ be a patch and $(u_0, v_0) \in U$. x is regular at (u_0, v_0) if $x_u \times x_v$ is nonzero at (u_0, v_0) . Thus we have following theorem.

Theorem 3.1. Regularity and singularity for the inverse surfaces are invariant properties under the inversion.

Proof. We assume that $\Phi = inv[c, r]$. Then, from [8] we have

$$y_u \times y_v = \lambda^2 \left(-x_u \times x_v + \frac{2 \langle (x - c), (x_u \times x_v) \rangle}{\|x - c\|^2} (x - c) \right). \quad (3.1)$$

If $p = x(u_0, v_0)$ a regular point of x , then

$$x_u \times x_v |_p \neq 0.$$

Assume that

$$y_u \times y_v |_q = 0,$$

where $\Phi(p) = q \in y(U)$. Thus from, (3.1) we can write

$$x_u \times x_v |_p = \frac{2 \langle (p - c), x_u \times x_v |_p \rangle}{\|p - c\|^2} (p - c). \quad (3.2)$$

Taking the inner product of (3.2) with $p - c$, we obtain

$$\langle x_u \times x_v |_p, p - c \rangle = 0.$$

If this last equation takes into account equality in (3.2), we have

$$x_u \times x_v |_p = 0.$$

So, the assumption is incorrect, that is, $\Phi(p) = q$ is the regular point of y .

Now, we assume that $p = x(u_0, v_0)$ is a singular point of x , then

$$x_u \times x_v |_p = 0.$$

Thus from the (3.1), we obtain that

$$y_u \times y_v |_q = 0.$$

This completes the proof.

Let I, II be the first and second fundamental forms of x , and let \tilde{I}, \tilde{II} be these of y . From [8], we have

$$\tilde{I} \circ \Phi_* = \lambda^2 I, \quad (3.3)$$

$$\widetilde{II} \circ \Phi_* = -\lambda II - 2\delta I, \quad (3.4)$$

where $\lambda(u, v) = \frac{r^2}{\|x(u, v) - c\|^2}$ and $\delta(u, v) = \frac{2r^2 \langle U_x(u, v), (x(u, v) - c) \rangle}{\|x(u, v) - c\|^4}$.

Thus, we have following results without proofs.

Corollary 3.2. ([1]) Let Γ_{jk}^i and $\widetilde{\Gamma}_{jk}^i$ ($i, j, k = 1, 2$) be Christoffel symbols of x and y , respectively. Then

$$\begin{aligned} \widetilde{\Gamma}_{11}^1 &= \Gamma_{11}^1 + \frac{[EG - 2F^2] \frac{\partial \lambda^2}{\partial u} + FE \frac{\partial \lambda^2}{\partial v}}{2\lambda^2 [EG - F^2]}, & \widetilde{\Gamma}_{11}^2 &= \Gamma_{11}^2 + \frac{EF \frac{\partial \lambda^2}{\partial u} - E^2 \frac{\partial \lambda^2}{\partial v}}{2\lambda^2 [EG - F^2]}, \\ \widetilde{\Gamma}_{12}^1 &= \Gamma_{12}^1 + \frac{GE \frac{\partial \lambda^2}{\partial v} - GF \frac{\partial \lambda^2}{\partial u}}{2\lambda^2 [EG - F^2]}, & \widetilde{\Gamma}_{12}^2 &= \Gamma_{12}^2 + \frac{EG \frac{\partial \lambda^2}{\partial u} - EF \frac{\partial \lambda^2}{\partial v}}{2\lambda^2 [EG - F^2]}, \\ \widetilde{\Gamma}_{22}^1 &= \Gamma_{22}^1 + \frac{GF \frac{\partial \lambda^2}{\partial v} - G^2 \frac{\partial \lambda^2}{\partial u}}{2\lambda^2 [EG - F^2]}, & \widetilde{\Gamma}_{22}^2 &= \Gamma_{22}^2 + \frac{[EG - 2F^2] \frac{\partial \lambda^2}{\partial v} + FG \frac{\partial \lambda^2}{\partial u}}{2\lambda^2 [EG - F^2]}, \end{aligned}$$

where $\lambda(u, v) = \frac{r^2}{\|x(u, v) - c\|^2}$ and E, F, G are coefficients of the first fundamental form of x .

Corollary 3.3. ([1]) Let Γ_{jk}^i and $\widetilde{\Gamma}_{jk}^i$ ($i, j, k = 1, 2$) be Christoffel symbols of x and y , respectively. Suppose that x is a principal patch in E^3 , then

$$\begin{aligned} \widetilde{\Gamma}_{11}^1 &= \Gamma_{11}^1 + (2\lambda)^{-2} \frac{\partial \lambda^2}{\partial u}, & \widetilde{\Gamma}_{11}^2 &= \Gamma_{11}^2 - (2\lambda)^{-2} \frac{E}{G} \frac{\partial \lambda^2}{\partial v}, \\ \widetilde{\Gamma}_{12}^1 &= \Gamma_{12}^1 + (2\lambda)^{-2} \frac{\partial \lambda^2}{\partial v}, & \widetilde{\Gamma}_{12}^2 &= \Gamma_{12}^2 + (2\lambda)^{-2} \frac{\partial \lambda^2}{\partial u} \\ \widetilde{\Gamma}_{22}^1 &= \Gamma_{22}^1 - (2\lambda)^{-2} \frac{G}{E} \frac{\partial \lambda^2}{\partial u}, & \widetilde{\Gamma}_{22}^2 &= \Gamma_{22}^2 + (2\lambda)^{-2} \frac{\partial \lambda^2}{\partial v} \end{aligned}$$

where $\lambda(u, v) = \frac{r^2}{\|x(u, v) - c\|^2}$ and E, G are coefficients of the first fundamental form of x .

Corollary 3.4 ([1]) Let S and \tilde{S} be the shape operators of x and y , respectively. Then

$$\tilde{S} \circ \Phi_* = -\lambda^{-1} S - \frac{2}{r^2} \eta I_2, \quad (3.8)$$

where I_2 is identity, $\lambda(u, v) = \frac{r^2}{\|x(u, v) - c\|^2}$ and $\eta = \langle U_x, (x - c) \rangle$.

Corollary 3.5. ([1]) (i) The inversion maps flat points into umbilic points,
(ii) The inversion maps flat points into flat points if and only if

$$\eta = \langle U_x, (x - c) \rangle = 0.$$

4. The Aymptotic and Conjugate Vectors in Inverse Surfaces

Lemma 4.1. Denote by III and \widetilde{III} the third fundamental forms of x and y , respectively. Then,

$$\widetilde{III} \circ \Phi_* = III + 2\xi II + \xi^2 I, \quad (4.1)$$

where $\xi = \frac{2\langle U_x, (x-c) \rangle}{\|x-c\|^2}$.

Proof. Assume that p be any point in $x(U)$ and $\Phi(p) = q \in y(U)$. Let the tangent map of Φ at p be $\Phi_{*p} = T_p(x) \longrightarrow T_{\Phi(p)}(y)$.

Suppose $u_p, v_p \in T_p(x)$ and $\tilde{u}_q, \tilde{v}_q \in T_q(y)$ such that

$$\Phi_{*p}(u_p) = \tilde{u}_q \quad \text{and} \quad \Phi_{*p}(v_p) = \tilde{v}_q.$$

Then

$$\widetilde{III}_q(\tilde{u}_q, \tilde{v}_q) = \left\langle \tilde{S}^2(\tilde{u}_q), \tilde{v}_q \right\rangle_q,$$

or

$$\widetilde{III}_q(\tilde{u}_q, \tilde{v}_q) = \left\langle \tilde{S}(\tilde{u}_q), \tilde{S}(\tilde{v}_q) \right\rangle_q. \quad (4.2)$$

Considering (3.3) and (3.8) in (4.2), we write

$$\widetilde{III}_q(\tilde{u}_q, \tilde{v}_q) = \lambda^2 \left\langle \left(-\lambda^{-1}S - \frac{2}{r^2}\eta I_2 \right)(u_p), \left(-\lambda^{-1}S - \frac{2}{r^2}\eta I_2 \right)(v_p) \right\rangle_p.$$

From above equation, we obtain.

$$\widetilde{III} \circ \Phi_* = III + 2\xi II + \xi^2 I.$$

Lemma 4.2. Let $\Phi_{*p} = T_p(x) \longrightarrow T_{\Phi(p)}(y)$. Then, the conjugate vectors in x are invariant under Φ_* if and only if

$$\eta = \langle U_x, (x-c) \rangle = 0$$

or these are perpendicular.

Proof. Let $p \in x(U)$ and $\Phi(p) = q \in y(U)$. We assume that

$$\Phi_*(u_p) = \tilde{u}_q \quad \text{and} \quad \Phi_*(v_p) = \tilde{v}_q,$$

where $u_p, v_p \in T_p(x)$ and $\tilde{u}_q, \tilde{v}_q \in T_q(y)$. Let u_p and $v_p \in T_p(x)$ be conjugate vectors. Then,

$$II_p(u_p, v_p) = 0. \quad (4.3)$$

By (3.4), we write that

$$\widetilde{II}_q(\tilde{u}_q, \tilde{v}_q) = \frac{-2r^2 \langle U_x |_p, (p-c) \rangle}{\|p-c\|^4} I_p(u_p, v_p), \quad (4.4)$$

where $\langle U_x |_p, (p - c) \rangle = \eta(p)$. If \tilde{u}_q and \tilde{v}_q are conjugate vectors then

$$\widetilde{II}_q(\tilde{u}_q, \tilde{v}_q) = 0.$$

Thus from (4.4), we obtain that

$$\eta(p) = \langle U_x |_p, (p - c) \rangle = 0,$$

or

$$I_p(u_p, v_p) = \langle u_p, v_p \rangle = 0.$$

Conversely, let $\eta(p) = 0$ or let $u_p \perp v_p$. Then, from (4.1), we write that

$$\widetilde{II}_q(\tilde{u}_q, \tilde{v}_q) = 0.$$

This implies that \tilde{u}_q and \tilde{v}_q are conjugate vectors. This completes the proof.

Theorem 4.3. Let III and \widetilde{III} be the third fundamental forms of x and y , respectively. Assume that $II \neq -\frac{\xi}{2}I$ and the conjugate vectors in x are not perpendicular each other, then following conditions are equivalent:

- (i) the conjugate vectors in x are invariant under Φ_*
- (ii) $\widetilde{III} \circ \Phi_* = III$,
- (iii) $U_y = -U_x$, where U_x and U_y are unit normals of x and y , respectively.

Proof. We show that $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (i)$. If (i) holds then

$$\eta = \langle U_x, (x - c) \rangle = 0,$$

thus $\xi = 0$. By (4.1), we obtain

$$\widetilde{III} \circ \Phi_* = III.$$

If (ii) holds then

$$\xi = \frac{2 \langle U_x, (x - c) \rangle}{\|x - c\|^2} = 0,$$

thus $\eta = 0$. Also, from [8], we write

$$U_y = -U_x + \frac{2\eta}{\|x - c\|^2} (x - c). \quad (4.5)$$

Hence, by (4.5), we obtain

$$U_y = -U_x.$$

Conversely, if (iii) holds then by (4.5)

$$\eta = 0,$$

thus it follows from Lemma 4.2 that (i) holds. This completes the proof.

Lemma 4.4. Let k and \tilde{k} be normal curvatures of x and y , respectively. Then

$$\tilde{k} \circ \Phi_* = -\frac{1}{\lambda}k - \frac{2}{r^2}\eta \quad (4.6)$$

where $\lambda(u, v) = \frac{r^2}{\|x(u, v) - c\|^2}$ and $\eta(u, v) = \langle U_x(u, v), (x(u, v) - c) \rangle$.

Proof. Let $p \in x(U)$, $\Phi(p) = q \in y(U)$ and $\Phi_{*p} = T_p(x) \longrightarrow T_{\Phi(p)}(y)$. Suppose that

$$\Phi_*(v_p) = \tilde{v}_q,$$

where $v_p \in T_p(x)$ and $\tilde{v}_q \in T_q(y)$. The normal curvature of y in the direction \tilde{v}_q is

$$\tilde{k}(\tilde{v}_q) = \frac{\widetilde{II}_q}{\widetilde{I}_q}(\tilde{v}_q, \tilde{v}_q). \quad (4.7)$$

On the other hand, from Lemma 2.2, we write

$$\|\tilde{v}_q\| = \lambda(p) \|v_p\|. \quad (4.8)$$

Thus, by (4.7) and (4.8), we obtain

$$\tilde{k} \circ \Phi_*(v_p) = -\frac{1}{\lambda(p)}k(v_p) - \frac{2}{r^2}\eta(p).$$

Lemma 4.5. Let $\Phi_{*p} = T_p(x) \longrightarrow T_{\Phi(p)}(y)$. Then, the asymptotic vectors in x are invariant under Φ_* if and only if

$$\eta = \langle U_x, (x - c) \rangle = 0.$$

Proof. Let $p \in x$, $\Phi(p) = q \in y$ and

$$\Phi_*(v_p) = \tilde{v}_q,$$

where $v_p \in T_p(x)$ and $\tilde{v}_q \in T_q(y)$. Suppose that $v_p \in T_p(x)$ is a asymptotic vector, then

$$k(v_p) = 0,$$

where k is the normal curvature of x . By Lemma 4.4, we can write

$$\tilde{k}(\tilde{v}_q) = -\frac{2}{r^2}\eta(p). \quad (4.9)$$

If asymptotic vectors in x are invariant under Φ_* , then also \tilde{v}_q is asymptotic and

$$\eta(p) = 0.$$

Conversely, if $\eta(p) = 0$ then, from (4.9) we obtain

$$\tilde{k}(\tilde{v}_q) = 0. \quad (4.10)$$

This is needed.

Theorem 4.6. Let $[III]$ and $[\widetilde{III}]$ be the third fundamental forms of x and y , respectively. Then following conditions are equivalent:

- (i) the asymptotic vectors in x are invariant under Φ_*
- (ii) $[\widetilde{III}] \circ \Phi_* = [III]$,
- (iii) $U_y = -U_x$, where U_x and U_y are unit normals of x and y , respectively.

Proof. The proof is the same with that of Theorem 4.3.

If the velocity vector of a curve is asymptotic vector, then the curve is asymptotic. Thus, we have a similar of Theorem 4.6 for asymptotic curves.

Now, we define the inverse curves of the parameter curves of a surface.

Definition 2.7. Let x be a patch in \mathbb{E}^{n*} and let y be the inverse patch of x with respect to Φ . Then, the curves

$$u \longrightarrow y(u, v_0) = c + \frac{r^2}{\|x(u, v_0) - c\|^2} (x(u, v_0) - c)$$

and

$$v \longrightarrow y(u_0, v) = c + \frac{r^2}{\|x(u_0, v) - c\|^2} (x(u_0, v) - c)$$

are called inverse parameter curves of u - parameter and v - parameter curves of x , respectively, where (u_0, v_0) is a constant point of U .

If parameter curves of a patch are asymptotic curves, then it is called asymptotic patch. Hence, a similar of theorem 4.6 for asymptotic patches can be given.

Theorem 4.8. Let τ and $\tilde{\tau}$ be torsions of the asymptotic curves on x and y , respectively. If

$$\eta = \langle U_x, (x - c) \rangle = 0$$

then

$$\tilde{\tau} = \pm \frac{1}{\lambda} \tau,$$

where $\lambda(u, v) = \frac{r^2}{\|x(u, v) - c\|^2}$.

Proof. If K and \tilde{K} respectively are Gauss curvatures of x and y , then from [8], we write

$$\tilde{K} = \frac{1}{\lambda^2}K + \frac{4}{r^2}\lambda^{-1}\eta H + \frac{4}{r^4}\eta^2,$$

where H is the mean curvature of x . Because $\eta = 0$, we have

$$\tilde{K} = \frac{1}{\lambda^2}K.$$

Since τ and $\tilde{\tau}$ are the torsions of the asymptotic curves in x and y , respectively, we obtain

$$\tilde{\tau} = \pm \frac{1}{\lambda}\tau.$$

5. Conclusions

For inverse surfaces, the minimality, the second and third fundamental forms, the shape operators, the normal curvatures, the conjugate and the asymptotic vectors and the torsions of the asymptotic curves are invariant properties under the inversion when $\eta = 0$. We comment the function η to be identically zero as follows: If

$$\eta(u, v) = \langle U_x(u, v), (x(u, v) - c) \rangle = 0,$$

then the tangent planes at all the points of the surface $x(u, v)$ pass through the center of the sphere of inversion. In such a case, the properties mentioned above are invariant under the inversion.

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